1. $\Delta u = \sin 2\varphi$, $r < 2$,
   $u(2,\varphi) = 3\cos 3\varphi$.
2. $\Delta u = 0$, $0 < x < \pi$, $0 < y < \pi$, если $u_x(0,y) = 2 + \cos y, u_x(\pi, y) = 2, u_y(x,0) = 0, u_y(x,\pi) = 0$
3. Решить $\Delta u = 0, -\infty < x < \infty, 0, < y < 2\pi; u(x,2\pi) = 0,$
   $u(x,0) = 1,$ при $x < 1; u(x,0) = 0,$ при $x \geq 1$
4. $$\begin{cases} u_{tt} = u_{xx} - \cos x, t > 0, 0 < x < \dfrac{\pi}{2}; \\ u_x(0,t) = 0, \quad u(\dfrac{\pi}{2}, t) = t, \\ u(x,0) = 2\cos 5x - \cos x, \\ u_t(x,0) = \cos x + 1. \end{cases}$$
5. $u_{tt} = 9u_{xx}, 0 < x < \infty, t > 0$

$u_x(0,t) = 0$, $u(x,0) = \begin{cases} \sin x, x \in [0;\pi] \\ 0, x \notin [0;\pi] \end{cases}$, $u_t(x,0) = 0$. Найти $u(x, \pi/2)$ и нарисовать график.

6. $u_{tt} = u_{xx}, t > 0, x > 0;$
   $u_x(0,t) = 0 \quad u(x,0) = \sin x \quad u_t(x,0) = \cos x.$

Описать процесс колебаний. Построить профиль струны в момент времени $t = \dfrac{3\pi}{4}$.

$$\begin{cases} \Delta u = \sin 2\varphi, r<2 \\ u(2,\varphi) = 3\cos 3\varphi \end{cases}$$

$u = R(r)\cdot \sin 2\varphi$

$\Delta u = (r^2 R'' + rR' - 4R)\sin 2\varphi = \sin 2\varphi$

$\Rightarrow r^2 R'' + rR' - 4R = r^2$

Решим однородное: (ур-ие Эйлера)

$\lambda(\lambda-1) + \lambda - 4 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = \pm 2 \Rightarrow R_{общ} = C_1 e^{2t} + C_2 e^{-2t}$

частное найдём в виде ate^{2t}

Характеристическое ур-ие: $y'' - 4y = e^{2t}$

$\Rightarrow a(e^{2t} + 2te^{2t})' - 4ate^{2t} = e^{2t}$

$\Leftrightarrow 2ae^{2t} + a\cdot 2e^{2t} + a\cdot 4te^{2t} - 4ate^{2t} = e^{2t}$

$\Leftrightarrow A = \frac{1}{4} \Rightarrow \frac{1}{4}te^{2t}$

$\Rightarrow u_r = \frac{\ln r \cdot r^2}{4}\sin 2\varphi$

$\exists\, u = u_1 + \frac{\ln r\cdot r^2}{4}\sin 2\varphi$

$\Delta u_1 = 0, \quad r<2$

$u(2,\varphi) = u_1(2,\varphi) + \frac{\ln 2\cdot 2^2}{4}\sin 2\varphi = u_1(2,\varphi) + \ln 2\cdot \sin 2\varphi$

$u_1(2,\varphi) = 3\cos 3\varphi - \ln 2 \sin 2\varphi$

Для круга:

$u(r,\varphi) = A + \sum r^n(A_n \cos n\varphi + B_n \sin n\varphi)$

$u(2,\varphi) = 3\cos 3\varphi - \ln 2 \sin 2\varphi$

$\Rightarrow A = 0,\ A_3 = \frac{3}{8},\ B_3 = 0$

$B_2 = -\frac{\ln 2}{4},\ A_2 = 0$

$u(r,t) = r^2\cdot\left(-\frac{\ln 2}{4}\sin 2\varphi\right) + r^3\cdot\frac{3}{8}\cdot\cos 3\varphi$

$\Rightarrow$ Ответ: $\frac{3}{8}r^3\cos 3\varphi - \frac{\ln 2}{4}r^2 \sin 2\varphi + \frac{\ln r\cdot r^2}{4}\sin 2\varphi$

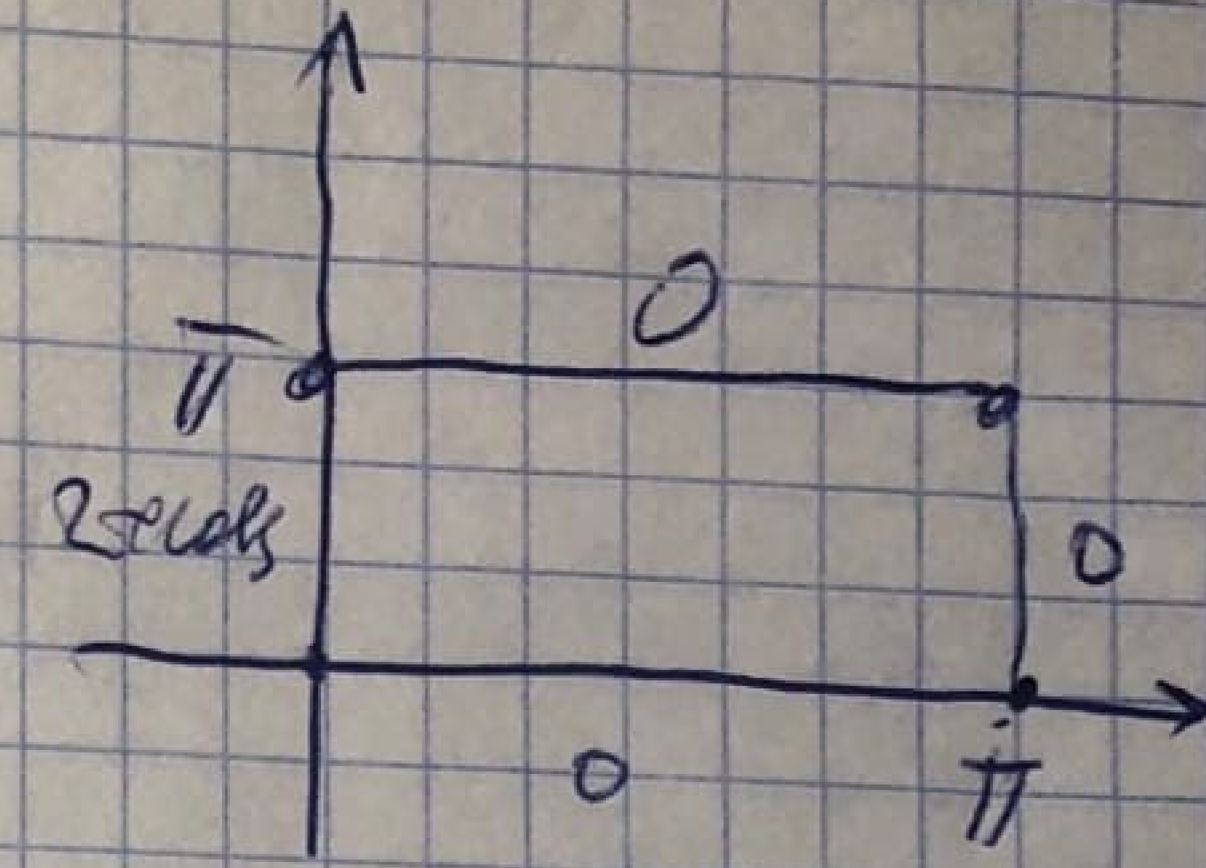
$\Delta u = 0$

$u_x(0, y) = 2 + \cos y$

$u_x(\pi, y) = 2$

$u_y(x, 0) = 0$

$u_y(x, \pi) = 0$



задача II–II

$\Rightarrow \lambda_n = n^2; n \in \mathbb{N}$; $y_n = \cos ny$

$\lambda_0 = 0$; $y_0 = 1$

$u(x,y) = A_0 + B_0 x + \sum (A_n e^{nx} + B_n e^{-nx}) \cos ny$

$u_x(x,y) = B + \sum (n e^{nx} A_n - n B_n e^{-nx}) \cos ny \Rightarrow$

$\Rightarrow u_x(0, y) = B + \sum n (A_n - B_n) \cdot \cos ny = 2 + \cos y$

$B = 2 \quad A_1 - B_1 = 1 \; ; \; A_n = B_n \quad n \neq 1.$

$u_x(\pi, y) = B + \sum n (e^{n\pi} A_n - e^{-n\pi} B_n) \cos ny = 2$

$e^{\pi} A_1 - e^{-\pi} B_1 = 0$; $A = B_n = 0$; $n \neq 1$

$\begin{cases} A_1 = e^{-2\pi} B_1 \\ B_1 (e^{-2\pi} - 1) = 1 \end{cases} \Rightarrow \begin{cases} B_1 = \dfrac{1}{e^{-2\pi} - 1} = \dfrac{e^{\pi}}{e^{-\pi} - e^{\pi}} = -\dfrac{e^{\pi}}{2 \operatorname{sh} \pi} \\ A_1 = \dfrac{e^{-2\pi}}{e^{-2\pi} - 1} = -\dfrac{e^{-\pi}}{2 \operatorname{sh} \pi} \end{cases}$

$\Rightarrow u(x,y) = A + 2x + \left(e^{x} \cdot \left(-\dfrac{e^{-\pi}}{2 \operatorname{sh} \pi} \right) + \dfrac{e^{\pi}}{2 \operatorname{sh} \pi} \cdot e^{-x} \right) \cos y$

№4.

$$\begin{cases} u_{tt} = u_{xx} - \cos x, \quad t>0, \quad 0<x<\frac{\pi}{2} \\ u_x(0,t)=0 \quad u(\frac{\pi}{2},t)=t \\ u(x,0) = 2\cos 5x - \cos x \\ u_t(x,0) = \cos x + 1 \end{cases}$$

$u(x,t) = U(x,t) + W(x,t)$, где $W(x,t) = t$

$$\begin{cases} V_{tt} = V_{xx} - \cos x, \quad t>0, \quad 0<x<\frac{\pi}{2} \\ U_x(0,t)=0 \quad U(\frac{\pi}{2},t)=0 \\ U(x,0) = 2\cos 5x - \cos x \\ u_t(x,0) = \cos x \end{cases}$$

1) $\begin{cases} U_{tt} = U_{xx} \\ -//- \end{cases}$

2) $\begin{cases} U_{tt} = U_{xx} - \cos x \\ U_x(0,t)=0 \quad U(\frac{\pi}{2},t)=0 \\ U(x,0)=0 \\ U_t(x,0)=\cos x \end{cases}$

1) $U = XT$

$T''X = X''T$

$\frac{T''}{T} = \frac{X''}{X} = -\lambda$

$\begin{cases} X'' + \lambda X = 0 \\ X'(0)=0 \\ X(\frac{\pi}{2})=0 \end{cases}$ II–I $\Rightarrow$ $\lambda_n = (2n+1)^2, \ n=0,\ldots$ $X_n = \cos(2n+1)x$

$T'' + \lambda T = 0$

$T = e^{kT}$

$k^2 + 1 = 0$

$\lambda > 0 \quad k = \pm i\sqrt{\lambda}$

$T_n = A_n \cos\sqrt{\lambda_n}\, t + B_n \sin\sqrt{\lambda_n}\, t$

$\upsilon(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos(2n+1)t + B_n \sin(2n+1)t\right) \cos(2n+1)x$

$\upsilon(x,0) = \sum_{n=0}^{\infty} A_n \cos(2n+1)x = 2\cos 5x - \cos x$

$\upsilon_t(x,0) = \sum_{n=0}^{\infty} (2n+1) B_n \cos(2n+1)x = \cos x$

$A_0 = -1 \quad A_2 = 2$

$B_0 = 1$

$\upsilon(x,t) = (-\cos t + \sin t)\cos x + 2\cos 5t \cos 5x$

2) $V(x,t) = \sum_{n=0}^{\infty} T_n X_n$

$$\begin{cases} \sum_{n=0}^{\infty} T_n'' \cos(2n+1)x = \sum_{n=0} T_n \cdot (-1)(2n+1)^2 \cos(2n+1)x - \cos x \\ \sum_{n=0}^{\infty} T_n(0) \cos(2n+1)x = 0 \\ \sum_{n=0}^{\infty} T_n'(0) \cos(2n+1)x = 0 \end{cases}$$

$$\begin{cases} T_0'' = -T_0 - 1 \\ T_0(0) = 0 \\ T_0'(0) = 0 \end{cases}$$

остальные 0

$$T_0 = \cos t - 1$$

Получаем $u = t + (\cos t - 1)\cos x + (\sin t - \cos t)\cos x +$

$+ 2\cos 5t \cos 5x = t + (\sin t - 1)\cos x + 2\cos 5t \cos 5x$

остальные 0

$\begin{cases} T_0'(0) = 0 \end{cases}$

$T_0 = \cos t - 1$

Получаем $u = t + (\cos t - 1)\cos x + (\sin t - \cos t)\cos x +$

$+ 2\cos 5t \cos 5x = t + (\sin t - 1)\cos x + 2\cos 5t \cos 5x$

№5.

$$\begin{cases} u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \ t > 0 \\ u_x(0,t) = 0 \qquad u(x,0) = \begin{cases} \sin x, & x \in [0,\pi] \\ 0, & x \notin [0,\pi] \end{cases} \quad u_t(x,0) = 0 \end{cases}$$

Найти $u(x, \frac{\pi}{2})$ и нарисовать график

Всп. задача на прямой (четный перенос)

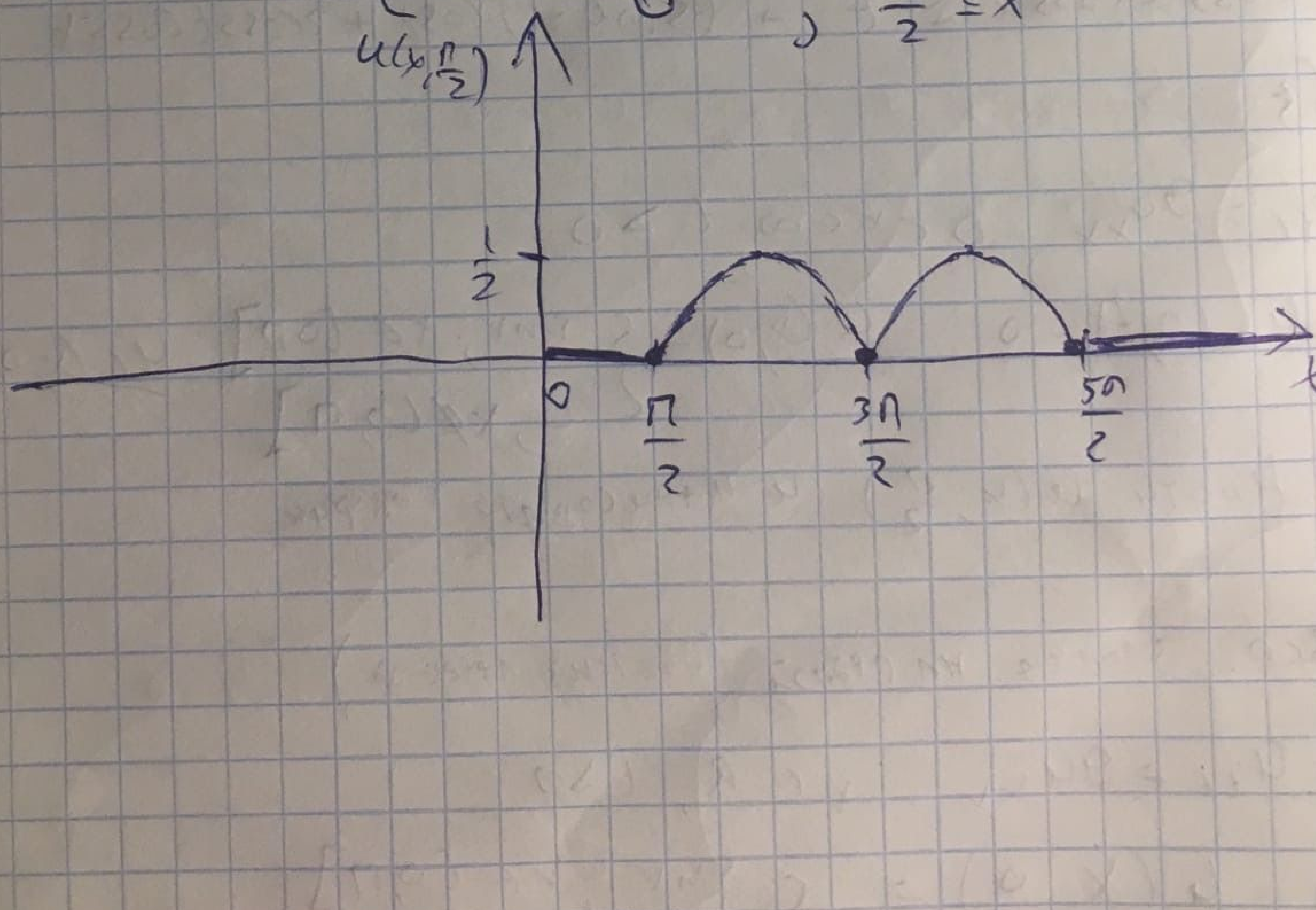
$$\begin{cases} u_{tt} = 9u_{xx}, \quad x \in R, \ t > 0 \\ u(x,0) = \begin{cases} \sin x, & x \in [0,\pi] \\ -\sin x, & x \in [-\pi, 0] \\ 0, & \text{иначе} \end{cases} = \varphi \\ u_t(x,0) = 0 = \psi \end{cases}$$

по ф-ле Даламбера: $u(x,t) = \dfrac{\varphi(x-3t) + \varphi(x+3t)}{2}$

$$u\left(x,\frac{\pi}{2}\right)=\frac{\varphi\left(x-\frac{3\pi}{2}\right)+\varphi\left(x+\frac{3\pi}{2}\right)}{2}$$

~~$\frac{\sin\left(x-\frac{3\pi}{2}\right)+\sin\left(x+\frac{3\pi}{2}\right)}{2}$, $0 \le x$~~

$$u\left(x,\frac{\pi}{2}\right)=\begin{cases} 0, & 0\le x<\frac{\pi}{2} \\ -\frac{\sin\left(x-\frac{3\pi}{2}\right)}{2}, & \frac{\pi}{2}\le x<\frac{3\pi}{2} \\ \frac{\sin\left(x-\frac{3\pi}{2}\right)}{2}, & \frac{3\pi}{2}< x<\frac{5\pi}{2} \\ 0, & \frac{5\pi}{2}\le x \end{cases}$$

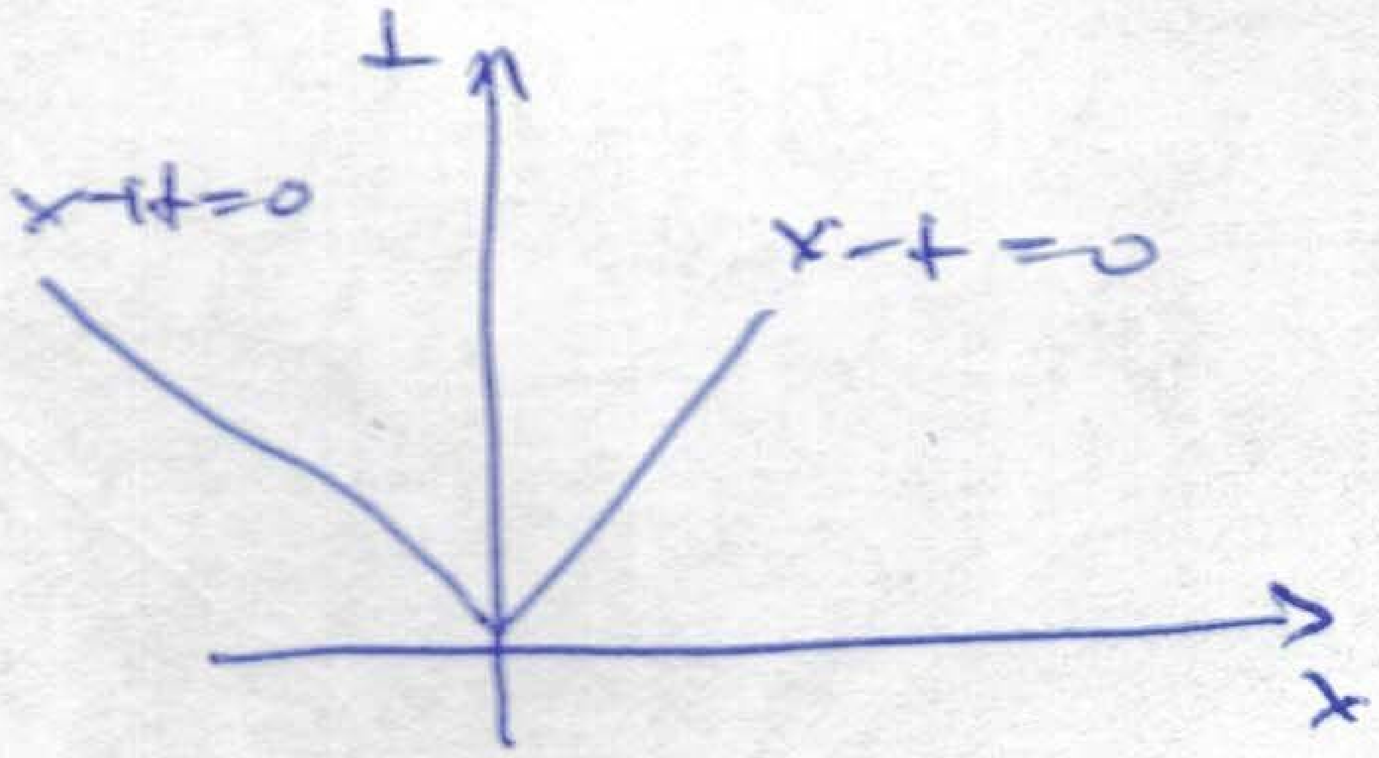


(6.1) Решим. вспомог. задачу

$$\begin{cases} u_{tt} = u_{xx}, \ t>0, \ x>0 \\ u(x,0) = \begin{cases} \sin x, & x>0 \\ -\sin x, & x<0 \end{cases} = \varphi(x) \\ u_t(x,0) = \cos x \end{cases}$$

$$u(x,t) = \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \cos x \, dx =$$

$$= \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2}(\sin(x+t) - \sin(x-t))$$



$$u(x,t) = \begin{cases} \frac{1}{2}(-\sin(x-t) + \sin(x+t) + \sin(x+t) - \sin(x-t)) = \sin(x+t) - \sin(x-t), & 0<x<t \\ \frac{1}{2}(\sin(x-t) + \sin(x-t) + \sin(x+t) - \sin(x-t)) = \sin(x+t), & x \geq t \end{cases}$$

$$u\left(x,\frac{\pi}{4}\right) = \begin{cases} \sin\left(x+\frac{\pi}{4}\right) - \sin\left(x-\frac{\pi}{4}\right) = 2\sin\frac{\pi}{4}\cos x = \sqrt{2}\cos x, & 0<x<\frac{\pi}{4} \\ \sin\left(x+\frac{\pi}{4}\right), & x \geq \frac{\pi}{4} \end{cases}$$